The Berkovich affine line $\mathbb{A}^1_{Berk}$

Sidi Mohammed Boughalem (shinokiz@gmail.com)

30. April 2018

We introduce here the Berkovich affine line $\mathbb{A}^1_{Berk}$ as a topological space, and shall see that it can be viewed as the direct limit (or union) of analytic discs of positive radii. We shall extend Berkovich's classification theorem to $\mathbb{A}^1_{Berk}$ and finally give a purely algebraic and intrinsic characterization of the different types of points in $\mathbb{A}^1_{Berk}$.

Definition 0.1. *As a topological space, we define $\mathbb{A}^1_{Berk}$ to be, as a set, the collection of all multiplicative seminorms $[.]_x$ on the polynomial ring $K[T]$ which extend the absolute value on K. The topology on $\mathbb{A}^1_{Berk}$ is the weakest one for which $x \mapsto [f]_x$ is continuous for all $f \in K[T]$.*

Let $R > 0$, we consider the Tate algebra

$$K\langle R^{-1}T\rangle = \{\sum_{k=0}^{\infty} c_k T^k \in K[[T]], R^k|c_k| \xrightarrow[k\to\infty]{} 0\}$$

We define the Berkovich disc of radius R as

$$\mathcal{D}(0,R) := \mathcal{M}(K\langle R^{-1}T\rangle)$$

Lemma 0.2.

$$\mathbb{A}^1_{Berk} \cong \bigcup_{R>0} \mathcal{D}(0,R)$$

Proof. Let $0 < r < R$, we consider the natural K-algebra homomorphism

$$\pi : K\langle R^{-1}T\rangle \to K\langle r^{-1}T\rangle$$

it induces a continuous map

$$i_{r,R} := \mathcal{M}(\pi) : \mathcal{D}(0,r) \to \mathcal{D}(0,R)$$

Seminar „Potential theory on the Berkovitch projective line “, WS 18/19, Universität Regensburg where for every $f \in K\langle R^{-1}T\rangle$

$$|f|_{i_{r,R}(x)} = |\pi(f)|_x$$

The map $i_{r,R}$ is injective, thus $\mathcal{D}(0,r)$ can be seen as a subspace of $\mathcal{D}(0,R)$ and it makes sense to consider the direct limit (or union) of $\mathcal{D}(0,R)$.

We shall prove this by specifying continuous maps in each direction which are inverse to one another.

(i) $\bigcup \mathcal{D}(0,R) \to \mathbb{A}^1_{Berk}$: Consider

$$\varphi : K[T] \hookrightarrow K\langle R^{-1}T\rangle$$

This induces -as above- a continuous map

$$i_R : \mathcal{D}(0,R) \hookrightarrow \mathbb{A}^1_{Berk}$$

On the other hand, for $r < R$

$$i_r : \mathcal{D}(0,r) \xrightarrow{i_{r,R}} \mathcal{D}(0,R) \xrightarrow{i_R} \mathbb{A}^1_{Berk}$$

Hence by the universal property of the direct limit, we get the continuous map

$$i : \varinjlim_{R>0} \mathcal{D}(0,R) \to \mathbb{A}^1_{Berk}$$

(ii) $\mathbb{A}^1_{Berk} \to \bigcup \mathcal{D}(0,R)$: Let $x \in \mathbb{A}^1_{Berk}$, $R = |T|_x$.

For $f = \sum_{i=0}^{\infty} c_i T^i \in K\langle R^{-1}T\rangle$, for $M \leq N$ we have

$$|[\sum_{i=0}^{N} c_i T^i]_x - [\sum_{i=0}^{M} c_i T^i]_x| \leq \max_{M \leq i \leq N} \{|c_i|R^i\}$$

Since $|c_i|R^i \xrightarrow[i\to\infty]{} 0$, $\{[\sum_{i=0}^{N} c_i T^i]_{x\}_{N \geq 0}}$ is a Cauchy sequence and therefor, by completness,

$$\lim_{N\to\infty} [\sum_{i=0}^{N} c_i T^i]_x \text{ exists.}$$

Now define

$$\psi : \mathbb{A}^1_{Berk} \to \bigcup_{R>0} \mathcal{D}(0,R)$$

where for every $f \in K\langle R^{-1}T\rangle$

$$|f|_{\psi(x)} = \lim_{N\to\infty} [\sum_{i=0}^{N} c_i T^i]_x$$

The defined maps are indeed inverse of one another. To see this we take at first $[.]_x \in \bigcup \mathcal{D}(0,R)$. This implies that $[.]_x \in \mathcal{D}(0,R)$ for a large $R >> 0$.

$$i([.]_x) = [.]_x \circ \varphi = [\varphi(.)]_x = [.]_{x'}$$

$$\Rightarrow \psi \circ i([.]_x) = \psi([.]_{x'}) \ \text{ where } \ [T]_{x'} = [\varphi(T)]_x = R$$

Hence

$$[f]_{\psi(x')} = \lim_{N\to\infty} [\sum_{i=0}^{N} c_i T^i]_{x'} = \lim_{N\to\infty} [\varphi(\sum_{i=0}^{N} c_i T^i)]_x = [f]_x$$

(since the $[.]_x$ are continuous)

Similarly, for $x \in \mathbb{A}^1_{Berk}$ we have that $\psi([.]_x) \in \mathcal{D}(0,R)$, hence $i([.]_{\psi(x)}) = [.]_{\psi(x)} \circ \varphi$. Let $\psi([.]_x) = [.]_{x''}$

$$i \circ \psi([.]_x) = i([.]_{x''}) = [\varphi(.)]_{x''}$$

It follows then that

$$[f]_{i\circ\psi(x)} = [\varphi(f)]_{x''} = \lim_{N\to\infty} [\sum_{i=0}^{N} c_i T^i]_x = [f]_x$$

□

Now consider $x \in \mathbb{A}^1_{Berk}$, we can enlarge R such that our x lands in a polydisc with radius in $|K^\times|$.

Let $a \in K$, $|a| = R$ and consider the following morphism $K\langle R^{-1}T\rangle \longrightarrow K\langle T\rangle$, $T \longmapsto aT$. This clearly induces a homeomorphism

$$\mathcal{D}(0,R) \xrightarrow{\sim} \mathcal{D}(0,1)$$

We can extend Berkovich's classification theorem to $\mathbb{A}^1_{Berk}$:

Theorem 0.3 (Berkovich's Classification theorem)**.** *Every $x \in \mathbb{A}^1_{Berk}$ can be realized as*

$$[f]_x = \lim_{i\to\infty} [f]_{D(a_i,r_i)}$$

for some sequence of nested closed discs $D(a_1,r_1) \supseteq D(a_2,r_2) \supseteq \ldots$ contained in K. If this sequence has a nonempty intersection, then either :

1. *the intersection is a single point a, in which case $[f]_x = |f(a)|$, or*
2. *the intersection is a closed disc $D(a,r)$ of radius $r > 0$, in which case $[f]_x = [f]_{D(a,r)}$.*

Thus, it makes sense to speak of points of types I, II, III, and IV in $\mathbb{A}^1_{Berk}$ corresponding to the seminorms associated to classical points, 'rational' discs $D(a,r)$ with radii in $|K^\times|$, 'irrational' discs $D(a,r)$ with radii not in $|K^\times|$, and nested sequences $\{D(a_i,r_i)\}$ with empty intersection, respectively. We will now give a more intrinsic characterization of the different types of points. For each x in $\mathbb{A}^1_{Berk}$ define its local ring in $K(T)$ by

$$\mathcal{R}_x = \{f = g/h \in K(T) : g,h \in K[T], [h]_x \neq 0\} \tag{1}$$

There is a natural extension of $[.]_x$ to a multiplicative seminorm on $\mathcal{R}_x$, given by

$$[g/h]_x = [g]_x/[h]_x$$

for g, h as in (1).

Let $[\mathcal{R}_x^\times]_x$ be the value group of $[.]_x$. Put

$$\mathcal{O}_x = \{f \in \mathcal{R}_x : [f]_x \leq 1\} \quad , \quad \mathfrak{m}_x = \{f \in \mathcal{R}_x : [f]_x < 1\}$$

and let $\tilde{k_x} = \mathcal{O}_x/\mathfrak{m}_x$ be the residue field. Recall that $\widetilde{K} = K^\circ/K^{\circ\circ}$ denotes the residue field of K. Recall also that we are assuming that K (and hence $\widetilde{K}$) is algebraically closed.

Proposition 0.4. *Given x in $\mathbb{A}^1_{Berk}$ we have the following :*

(A) x is of type I $\Leftrightarrow \mathcal{R}_x \subsetneq K(T)$, $[\mathcal{R}_x^\times]_x = |K^\times|$, and $\tilde{k_x} = \widetilde{K}$.

(B) x is of type II $\Leftrightarrow \mathcal{R}_x = K(T)$, $[\mathcal{R}_x^\times]_x = |K^\times|$, and $\tilde{k_x} = \widetilde{K}(t)$, t transcendental over $\widetilde{K}$.

(C) x is of type III $\Leftrightarrow \mathcal{R}_x = K(T)$, $[\mathcal{R}_x^\times]_x \supsetneq |K^\times|$, and $\tilde{k_x} = \widetilde{K}$.

(D) x is of type IV $\Leftrightarrow \mathcal{R}_x = K(T)$, $[\mathcal{R}_x^\times]_x = |K^\times|$, and $\tilde{k_x} = \widetilde{K}$.

Proof. The possibilities for the triples $(\mathcal{R}_x, [\mathcal{R}_x^\times]_x, \tilde{k_x})$ are mutually exclusive, so it suffices to prove all implications in the forward direction. Note that we always have $|K^\times| \subset [\mathcal{R}_x^\times]_x$

(A) Let x be of type I, i.e. $x \rightsquigarrow a \in K$ and $[f]_x = |f(a)|$.

- $\mathcal{R}_x \subsetneq K(T)$: Indeed, we can see that $\mathfrak{m}_a = T - a \in K(T) \setminus \mathcal{R}_x$.
- $[\mathcal{R}_x^\times]_x = |K^\times|$: Comes clearly from the fact that $[f]_x = |f(a)| \in |K^\times|$
- $\tilde{k_x} \cong \widetilde{K}$: Consider

$$\alpha : \mathcal{O}_x \to K^\circ \twoheadrightarrow \widetilde{K} = K^\circ/K^{\circ\circ}$$
$$f \mapsto f(a) \mapsto f(a) + K^{\circ\circ}$$

α is surjective since $\mathcal{O}_x \to K^\circ$ is ($K^\circ \subset \mathcal{O}_x$) hence we get an isomorphism

$$\mathcal{O}/\ker\alpha \cong \widetilde{K}$$

where $\ker\alpha = \{f \in \mathcal{O}_x, f(a) \in K^{\circ\circ}\} = \{f \in \mathcal{R}_x : [f]_x < 1\} = \mathfrak{m}_x$

(B) Let x be of type II, i.e. $x \rightsquigarrow D(a,r)$ with $r \in |K^\times|$.

- $\mathcal{R}_x = K(T)$: This is due to the fact that no nonzero polynomial can vanish identically on $D(a,r)$, in fact for $h = \sum b_i(T-a)^i$ we have

$$[h]_x = \max_i |b_i| r^i = 0 \Leftrightarrow \forall i : b_i = 0 \Leftrightarrow h = 0.$$

- $[\mathcal{R}_x^\times]_x = |K^\times|$: Let $g \in K[T]$, the non-Archimedean maximum principle (see ([2] Proposition 3, p. 197) states that

$$[f]_x = \max_{x \in D(a,r)} |f(x)|$$

Hence, there is a point $p \in D(a,r)$ where the maximum is attained, and for which $[g]_x = |g(p)| \in |K^\times|$, from which follows the wanted result.

- $\tilde{k_x} = \widetilde{K}(t)$: Let $c \in K^\times$ satisfy $|c| = r$, and let $t \in \tilde{k_x}$ be the reduction of $(T-a)/c$. We claim that t is transcendental over $\widetilde{K}$.

Suppose that t is algebraic over $\widetilde{K} \Leftrightarrow \exists \tilde{P} \in \widetilde{K}[T]$, non zero, such that $\tilde{P}(t) = 0$ Let $n > 0$ be minimal such that

$$t^n + \tilde{a}_{n-1} t^{n-1} + ... + \tilde{a}_0 = 0$$

with $\tilde{a}_{n-1}, ..., \tilde{a}_0 \in \widetilde{K}$. Let a_i be preimages of $\tilde{a}_i$ as follow :

$$\pi : K^\circ \twoheadrightarrow \widetilde{K}, \quad \pi(a_i) = \tilde{a}_i, \quad \pi(\frac{T-a}{c}) = t$$

and

$$P = \sum_{i=1}^{n} a_i (\frac{T-a}{c})^i = \sum_i \frac{a_i}{c^i}(T-a)^i$$

In particular,

$$[(\frac{T-a}{c})^n + a_{n-1}(\frac{T-a}{c})^{n-1} + ... + a_0]_x < 1 \text{ in } \tilde{k_x} \tag{2}$$

But, since we know that

$$[f]_x = \max_{b \in D(a,r)} |f(b)| \Rightarrow [P]_x = [P]_{D(a,r)} = \max_i \frac{|a_i|}{|c^i|} r^i = \max_i |a_i| < 1$$

Hence we get $|\tilde{a}_i| = 0$ in $\widetilde{K}$ $\forall i$ which is impossible.

(C) Let x be of type III, i.e. $x \rightsquigarrow D(a,r)$ with $r \notin |K^\times|$.

- $\mathcal{R}_x = K(T)$: By the same argument as in (B).
- $[\mathcal{R}_x^\times]_x \supsetneq |K^\times|$: Indeed, $[T-a]_x = r \notin |K^\times|$ so $|K^\times| \subsetneq [\mathcal{R}_x^\times]_x =< |K^\times|, r >$

– $\tilde{k_x} = \widetilde{K}$: Let $f = g/h \in K(T)$. We claim that f has a constant reduction for certain polynomials g, h and $[f]_x \leq 1$. Indeed, let

$$g(T) = \sum_{i=0}^{m} b_i(T-a)^i \quad h(T) = \sum_{j=0}^{n} c_j(T-a)^j$$

Note that for $i \neq j$: $|b_i|_x|T-a|_x^j \neq |b_j|_x|T-a|_x^j$ (since one factor is in the value group and the other is not) hence we get by the ultrametric inequality (Lemma 1.1)

$$[g(T) = \sum_{i=0}^{m} b_i(T-a)^i]_x = \max_{0\leq i\leq n} [bi]_x r^i = [b_{i_0}]_x r^{i_0}$$

similarly, there exist a j_0 index such that

$$[h(T)]_x = [c_{j_0}]_x r^{j_0}$$

Now if $[f]_x = [g]_x/[h]_x = 1$, we have that

$$|b_{i_0}|r^{i_0} = |c_{j_0}|r^{j_0} \Leftrightarrow i_0 = j_0$$

Now we have that

$$|b_i|r^i \leq [h]_x(= [g]_x) \Rightarrow \frac{|b_i|r^i}{[h]_x} \leq 1 \Rightarrow \frac{|b_{i_0}|r^{i_0}}{[h]_x} = 1 \text{ and } \frac{|b_i|r^i}{[h]_x} < 1 \text{ for } i \neq i_0$$

Hence, by passing modulo $\mathfrak{m}_x$

$$f = \sum_{i=0}^{m} \frac{b_i(T-a)^i}{h} \equiv \frac{b_{i_0}(T-a)^i}{h} \mod \mathfrak{m}_x$$

By the same process on $f' = h/g$ we obtain

$$\frac{|c_{i_0}|r^{i_0}}{[g]_x} = 1 \text{ and } \frac{|c_i|r^i}{[g]_x} < 1 \text{ for } i \neq i_0$$

and thus finally

$$f \equiv \frac{b_{i_0}(T-a)^i}{c_{i_0}(T-a)^i} \equiv \frac{b_{i_0}}{c_{i_0}} \mod \mathfrak{m}_x$$

(D) Let x be of type IV, i.e. $x \rightsquigarrow$ a nested sequence of discs $\{D(a_i, r_i)\}$ with empty intersection.

– $|K^\times| = [\mathcal{R}_x^\times]_x$: For each n where $D(a_n, r_n)$ contains zeros of $g \in K[T]$, $\exists N$ such that $D(a_N, r_N)$ does not contain any of the zeros of g (the discs have empty intersection).

Let $b_1, ..., b_k \notin D(a_N, r_N)$ be zeros of $g = c(T - b_1)...(T - b_k)$ and $y \in D(a_N, r_N) \subset K$.

We have that

$$[g]_y = [c(T - a_N + a_N - b_1)...(T - a_N + a_N - b_k)]_y$$

Since $[T - a_N]_y \leq r_N$ and $[a_N - b_i]_y > r_N$ we have that $[T - a_N + a_N - b_i]_y = [a_N - b_i]_y$ and thus

$$[g]_y = |g(y)| = |c(a_N - b_1)...(a_N - b_k)| \text{ is constant}$$

and finally

$$[g]_x = \inf_i [g]_{D(a_i,r_i)} = [g]_{D(a_N,r_N)}$$

x can not be of type I since the nested sequence has empty intersection, it is then of type II or III, if we have points of type III, we can always find a point of type II simply by taking an r'' between two r and r' of points of type III.

- $\mathcal{R}_x = K(T)$: From (B) and (C) we have also that $\mathcal{R}_x = K(T)$.
- $\widetilde{k_x} = \widetilde{K}$: Now as done in (C), let $f = g/h \in K(T)$ such that $[f]_x = 1$. There exists an N for which $[g]_x = [g]_{D(a_N,r_N)}$ and $[h]_x = [h]_D(a_N, r_N)$, we get necessarily that $[g]_x = [h]_x$ from ([1] Corollary A.19 in Appendix A) we have that $|g(z) - g(a_N)| < |g(a_N)|$ and since

$$[g - g(a_N)]_x = \inf_i [g - g(a_N)]_{D(a_N,r_N)} = \max_{z \in D(a_N,r_N)} |g(z) - g(a_N)| < |g(a_N)| = [g]_{D(a_N,r_N)} = |g|_x$$

(since $|g(z)|$ is constant on $D(a_N, r_N)$) Similarly, we get that

$$|h - h(a_N)|_x < |h|_x$$

Hence we get

$$f = \frac{g}{h} = \frac{g + g(a_N) - g(a_N)}{h} = \frac{g - g(a_N)}{h} + \frac{g(a_N)}{h}$$

Since $[g]_x = [h]_x$ and $[\frac{g-g(a_N)}{h}]_x < [\frac{g}{h}]_x = 1$ we get that

$$f \equiv \frac{g(a_N)}{h} \mod \mathfrak{m}_x$$

By doing the same procedure (again) on $f' = h/g$ we get that $[\frac{h-h(a_N)}{g}]_x < 1$ and thus

$$f \equiv \frac{g(a_N)}{h(a_N)} \mod \mathfrak{m}_x$$

$\square$

References

[1] M. Baker, R. Rumely *Potential theory and dynamics on the Berkovich projective line*, Mathematical Surveys and Monographs, 159. American Mathematical Society, Providence, RI, 2010.

[2] S. Bosch, U. Güntzer, R. Remmert *Non-Archimedean analysis*, volume 261 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1984.